The normalized energy eigenspinors of the Dirac field on anti-de Sitter spacetime

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Abstract

It is shown how can be derived the normalized energy eigenspinors of the free Dirac field on anti-de Sitter spacetime, by using a Cartesian tetrad gauge where the separation of spherical variables can be done like in special relativity.

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A fundamental but difficult problem in general relativity is to find the analytical forms of the quantum free fields in given local charts. The difficulties arise because the particular solutions of the field equations that describe the one-particle quantum modes are strongly dependent on the procedure of separation of variables and, implicitly, on the choice of the holonomic coordinates. For the Dirac field the situation is more complicated since the form of the field equations depends, in addition, on the tetrad gauge in which one works. In these conditions it is helpful to exploit the effects of the global symmetries of background that guarantees the existence of conserved observables.

An important case is that of the Dirac equation on spacetimes with spherically symmetric (central) static charts that have the global symmetry of the group $T(1)\otimes SO(3)$, of time translations and rotations of the Cartesian space coordinates. For these backgrounds we have proposed recently a Cartesian

tetrad gauge [1, 2] which assures the covariance under this group of the Dirac equation in Cartesian coordinates, the whole theory acquiring thus the global symmetry of the background. Consequently, the energy and angular momentum are conserved like in special relativity from which we can take over the method of separation of variables in spherical coordinates. In fact, our gauge defines Cartesian local (unholonomic) frames which play the same role as the Cartesian natural frame of the Minkowski spacetime, since their axes are just those of projections of the orbital angular momentum. This allowed us to separate the spherical variables in terms of usual angular spinors such that all the constants involved in separation of variables get physical meaning as eigenvalues of the familiar set of commuting operators H, J^2, J_3 and K [3, 4]. On this way we have found a complete formulation of the radial problem of the Dirac equation on central static backgrounds, deriving the radial equations and the form of the radial scalar product in the general case [1, 2]. Similar results have been obtained in Ref.[5] by using a different method based on an extended gauge.

In this approach the Dirac equation on anti-de Sitter (AdS) or de-Sitter static backgrounds can be analytically solved in terms of Gauss hypergeometric functions. Thus we have obtained the energy spectra and the energy eigenspinors up to normalization factors, in both these cases [2, 6]. However, the normalization of the energy eigenspinors is very important for the further developments of the quantum theory. In the de Sitter spacetime, where the energy spectrum is continuous, it is less probable to find an efficient normalization procedure but on AdS the energy spectrum is discrete and, therefore, the normalization may be done in usual way. This is just the problem we would like to discuss here. Our aim is to present how can be selected the quantum modes and to write down the normalized energy eigenspinors of the regular modes of the Dirac field on AdS, in spherical coordinates (and natural units with $\hbar = c = 1$). Moreover, we establish their orthogonality properties with respect to the relativistic scalar product. In order to introduce the framework we need, we start with a brief review of some general results concerning the radial Dirac problem in our gauge and then turn to the AdS case.

Let us denote by ψ the free Dirac field of mass M defined on a central static chart with Cartesian coordinates $(t, \mathbf{x})$ or spherical natural coordinates (t, r, θ, ϕ), commonly related to the Cartesian ones. The space domain of this chart is $D = D_r \times S^2$, i.e. $r \in D_r$ while θ and ϕ cover the sphere S^2. In the

following it is useful to consider only the charts where the radial coordinate $r = |\mathbf{x}|$ is defined such that $g_{rr} = -g_{00}$. Then the tetrad fields in our Cartesian gauge [1, 2] depend on two arbitrary functions of r, denoted by v and w, which give the line element

$$ds^2 = w^2 \left[dt^2 - dr^2 - \frac{r^2}{v^2}(d\theta^2 + \sin^2\theta d\phi^2) \right] . \tag{1}$$

and determine a suitable form of the Dirac equation. This has particular positive frequency solutions of energy E,

$$\psi^{(+)}_{E,j,m_j,\kappa_j}(t, \mathbf{x}) = U_{E,j,m_j,\kappa_j}(\mathbf{x}) e^{-iEt} , \tag{2}$$

where

$$\begin{aligned} U_{E,j,m_j,\kappa_j}(\mathbf{x}) &= U_{E,j,m_j,\kappa_j}(r, \theta, \phi) \\ &= \frac{v(r)}{rw(r)^{3/2}} [f^+(r)\Phi^+_{m_j,\kappa_j}(\theta, \phi) + f^-(r)\Phi^-_{m_j,\kappa_j}(\theta, \phi)] \end{aligned} \tag{3}$$

are the particle-like energy eigenspinors expressed in terms of radial wave functions $f^\pm$ and four-component angular spinors $\Phi^\pm_{m_j,\kappa_j}$, known from special relativity [4]. We remind that they are orthogonal to each other being completely determined by the angular quantum numbers, j and m_j, and the value of $\kappa_j = \pm(j + 1/2)$ [4, 3]. Moreover, they are normalized to unity with respect to their own angular scalar product. Thus the problem of the angular motion is solved in the same manner as in the flat spacetime. Obviously, the radial problem is different. The radial functions $f^\pm$ are solutions of a pair of radial equations [1, 2] that can be written in compact form as the eigenvalue problem

$$H\mathcal{F} = E\mathcal{F} \tag{4}$$

of the radial Hamiltonian

$$H = \left| \begin{array}{cc} Mw & -\frac{d}{dr} + \kappa_j \frac{v}{r} \\ & \\ \frac{d}{dr} + \kappa_j \frac{v}{r} & -Mw \end{array} \right| , \tag{5}$$

in the space of two-component vectors, $\mathcal{F} = |f^+, f^-|^T$, where the radial scalar product is [1, 2]

$$(\mathcal{F}_1, \mathcal{F}_2) = \int_{D_r} dr\, \mathcal{F}_1^\dagger \mathcal{F}_2 . \tag{6}$$

This selects the "good" radial wave functions (i.e. square integrable functions or tempered distributions) which enter in the structure of the particle-like energy eigenspinors (3). We note that the antiparticle-like energy eigenspinors can be obtained directly by using the charge conjugation [2].

In the case of AdS spacetime it is convenient to consider the central static chart with the line element [7]

$$ds^2 = \sec^2 \omega r \left[dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r \ (d\theta^2 + \sin^2 \theta \ d\phi^2) \right]. \tag{7}$$

The radial domain of this chart is $D_r = [0, \pi/2\omega)$ because of the event horizon at $r = \pi/2\omega$. We specify that here we take $t \in (-\infty, \infty)$ which defines in fact the universal covering spacetime (CAdS) of AdS [7]. Now, from (7) we can identify the functions $w(r) = \sec \omega r$ and $v(r) = \omega r \csc \omega r$. With their help and by using the notation $k = M/\omega$ (i.e. $Mc^2/\hbar\omega$ in usual units), we obtain

$$H = \left| \begin{array}{cc} \omega k \sec \omega r & -\frac{d}{dr} + \omega \kappa_j \csc \omega r \\ & \\ \frac{d}{dr} + \omega \kappa_j \csc \omega r & -\omega k \sec \omega r \end{array} \right|. \tag{8}$$

This Hamiltonian has a hidden supersymmetry that can be pointed out with the help of the local rotation $\mathcal{F} \to \hat{\mathcal{F}} = R\mathcal{F} = |\hat{f}^+, \hat{f}^-|^T$ produced by

$$R(r) = \left| \begin{array}{cc} \cos \frac{\omega r}{2} & -\sin \frac{\omega r}{2} \\ & \\ \sin \frac{\omega r}{2} & \cos \frac{\omega r}{2} \end{array} \right|. \tag{9}$$

Indeed, after a few manipulation we find that the transformed (i.e. rotated and translated) Hamiltonian,

$$\hat{H} = RHR^T - \frac{\omega}{2} 1_{2\times 2}, \tag{10}$$

has supersymmetry since it has the requested specific form with diagonal constant terms [4] and a Pöschl-Teller-like [8] superpotential [2].

In these conditions the new eigenvalue problem

$$\hat{H}\hat{\mathcal{F}} = \left(E - \frac{\omega}{2} \right) \hat{\mathcal{F}}, \tag{11}$$

involving the transformed radial wave functions $\hat{f}^{\pm}$, leads to a pair of second order equations giving [2]

$$\begin{aligned}\hat{f}^{\pm}(r) &= N_{\pm}\sin^{2s_{\pm}}\omega r\cos^{2p_{\pm}}\omega r \\ &\quad\times F\left(s_{\pm}+p_{\pm}-\frac{\epsilon}{2}, s_{\pm}+p_{\pm}+\frac{\epsilon}{2}, 2s_{\pm}+\frac{1}{2}, \sin^{2}\omega r\right).\end{aligned} \tag{12}$$

where F are the Gauss hypergeometric functions [9]. Their real parameters are defined as, $\epsilon = E/\omega - 1/2$ and

$$2s_{\pm}(2s_{\pm}-1) = \kappa_j(\kappa_j \pm 1)\,, \tag{13}$$

$$2p_{\pm}(2p_{\pm}-1) = k(k \mp 1)\,, \tag{14}$$

while $N_{\pm}$ are normalization factors. The next step is to select the suitable values of these parameters and to calculate N_+/N_- such that the functions $\hat{f}^{\pm}$ should be solutions of the transformed radial problem (11), with a good physical meaning. This can be achieved only when F is a polynomial selected by a suitable quantization condition since otherwise F is strongly divergent for $\sin^2\omega r \to 1$. Then the functions $\hat{f}^{\pm}$ will be square integrable with normalization factors calculated according to the condition

$$(\mathcal{F},\mathcal{F}) = (\hat{\mathcal{F}},\hat{\mathcal{F}}) = \int_{D_r} dr\,\left(|\hat{f}^{+}(r)|^2 + |\hat{f}^{-}(r)|^2\right) = 1\,, \tag{15}$$

resulted from the fact that the matrix (9) is orthogonal.

The discrete energy spectrum is given by the particle-like CAdS quantization conditions

$$\epsilon = 2(n_{\pm}+s_{\pm}+p_{\pm})\,, \quad \epsilon > 0\,, \tag{16}$$

that must be compatible with each other, i.e.

$$n_+ + s_+ + p_+ = n_- + s_- + p_-. \tag{17}$$

Hereby we see that there is only one independent *radial* quantum number, $n_r = 0, 1, 2, ...$ In addition, we shall use the orbital quantum number l of the spinor $\Phi^{+}_{m_j,\kappa_j}$ [3], as an auxiliary quantum number. On the other hand, if we express (12) in terms of Jacobi polynomials, we observe that these functions remain square integrable for $2s_{\pm} > -1/2$ and $2p_{\pm} > -1/2$. Since $l = 0, 1, 2...$ we are forced to select only the positive solutions of Eqs.(13). The different

solutions of Eqs.(14) defines the boundary conditions of the allowed quantum modes, like in the case of the scalar modes [10]. We say that for $k > -1/2$ the values $2p_+ = k$ and $2p_- = k+1$ define the boundary conditions of *regular* modes. The other possible values, $2p_+ = -k+1$ and $2p_- = -k$, define the *irregular* modes when $k < 1/2$. Obviously, for $-1/2 < k < 1/2$ both these modes are possible. We note that the AdS quantization conditions require, in addition, k to be a half integer. Then it is clear that the domains of k corresponding to the regular and respectively irregular modes can not overlap with each other. Anyway, in our opinion, the problem of the meaning of the irregular modes as well as that of the relation between these kind of modes is sensitive and may be carefully analyzed. For this reason we restrict ourselves to write down only the energy eigenspinors of the regular modes on CAdS.

Let us take first $\kappa_j = -(j+1/2) = -l-1$. Then the positive solutions of (13) are $2s_+ = l+1$ and $2s_- = l+2$ while, according to (17), we must have $n_+ = n_r$ and $n_- = n_r - 1$. For these values of parameters, the functions $\hat{f}^{\pm}$ given by (12) and (16) represent a solution of the transformed radial problem (11) if and only if

$$\frac{N_-}{N_+} = -\frac{2n_r}{2l+3}. \tag{18}$$

Furthermore, it is easy to express (12) in terms of Jacobi polynomials and to calculate the normalization factors from (15). Thus we arrive at the result,

$$\begin{aligned} \hat{f}^+(r)_{|\kappa_j=-(j+1/2)} &= N\left[\frac{n_r+k+l+1}{n_r+l+\frac{1}{2}}\right]^{\frac{1}{2}} \\ &\quad \times \sin^{l+1}\omega r \cos^k \omega r P_{n_r}^{(l+\frac{1}{2},k-\frac{1}{2})}(\cos 2\omega r)\,, \\ \hat{f}^-(r)_{|\kappa_j=-(j+1/2)} &= -N\left[\frac{n_r+k+l+1}{n_r+l+\frac{1}{2}}\right]^{\frac{1}{2}} \\ &\quad \times \sin^{l+2}\omega r \cos^{k+1} \omega r P_{n_r-1}^{(l+\frac{3}{2},k+\frac{1}{2})}(\cos 2\omega r)\,, \end{aligned} \tag{19}$$

where

$$N = \eta\sqrt{2\omega}\left[\frac{n_r!\,\Gamma(n_r+k+l+1)}{\Gamma(n_r+l+\frac{1}{2})\Gamma(n_r+k+\frac{1}{2})}\right]^{\frac{1}{2}}. \tag{20}$$

is defined up to the phase factor η. Notice that from (18) we understand that the second equation of (19) gives $\hat{f}^- = 0$ for $n_r = 0$.

For $\kappa_j = j + 1/2 = l$ we use the same procedure finding that $2s_+ = l + 1$, $2s_- = l$, $n_+ = n_- = n_r$ and

$$\frac{N_-}{N_+} = \frac{2l+1}{2n_r + 2k + 1}\,. \tag{21}$$

In this case the normalized radial wave functions are

$$\begin{aligned}
\hat{f}^+(r)_{|\kappa_j=j+1/2} &= N\left[\frac{n_r + k + \frac{1}{2}}{n_r + l + \frac{1}{2}}\right]^{\frac{1}{2}} \\
&\quad \times \sin^{l+1}\omega r \cos^k \omega r P_{n_r}^{(l+\frac{1}{2},k-\frac{1}{2})}(\cos 2\omega r)\,, \qquad (22)\\
\hat{f}^-(r)_{|\kappa_j=j+1/2} &= N\left[\frac{n_r + l + \frac{1}{2}}{n_r + k + \frac{1}{2}}\right]^{\frac{1}{2}} \\
&\quad \times \sin^{l}\omega r \cos^{k+1} \omega r P_{n_r}^{(l-\frac{1}{2},k+\frac{1}{2})}(\cos 2\omega r)\,.
\end{aligned}$$

The energy levels result from (16). Bearing in mind that $\omega k = M$ and $\omega\epsilon = E - \omega/2$, and by using the *main* quantum number $n = 2n_r + l$ we obtain [2]

$$E_n = M + \omega\left(n + \frac{3}{2}\right)\,, \quad n = 0, 1, 2, ... \tag{23}$$

These levels are degenerated. For a given n our auxiliary quantum number l takes either all the odd values from 1 to n, if n is odd, or the even values from 0 to n, if n is even. In both cases we have $j = l \pm 1/2$ for each l, which means that $j = 1/2, 3/2, ..., n + 1/2$. The selection rule for κ_j is more complicated since it is determined by both the quantum numbers n and j. If n is even then the even κ_j are positive while the odd κ_j are negative. For odd n we are in the opposite situation, with odd positive or even negative values of κ_j. Thus it is clear that for each given pair (n, j) we have only one value of κ_j. With these specifications and by taking into account that for each j we have $2j + 1$ different values of m_j, we can conclude that the degree of degeneracy of the level E_n is $(n + 1)(n + 2)$.

The solutions (19) and (22) are completely determined by the values of n and j since, according to the above rules, we have

$$2n_r = n - j - \frac{1}{2}(-1)^{n+j+1/2}\,, \quad l = j + \frac{1}{2}(-1)^{n+j+1/2}\,. \tag{24}$$

For this reason, we denote by $\hat{f}^{\pm}_{n,j}$ the radial wave functions (19) and (22). With their help we can write the functions $f^{\pm}_{n,j}$ (i.e. the components of $\mathcal{F}$) by using the inverse of (9). Then from (3) we find the definitive form of the normalized particle-like energy eigenspinors of the regular modes,

$$\begin{aligned} U_{n,j,m_j}(\mathbf{x}) &= \omega \csc \omega r \cos^{3/2} \omega r \\ &\times \left[\left(\cos \frac{\omega r}{2} \hat{f}^{+}_{n,j}(r) + \sin \frac{\omega r}{2} \hat{f}^{-}_{n,j}(r) \right) \Phi^{+}_{m_j,\kappa_j}(\theta,\phi) \right. \\ &+ \left. \left(-\sin \frac{\omega r}{2} \hat{f}^{+}_{n,j}(r) + \cos \frac{\omega r}{2} \hat{f}^{-}_{n,j}(r) \right) \Phi^{-}_{m_j,\kappa_j}(\theta,\phi) \right]. \end{aligned} \tag{25}$$

The antiparticle-like energy eigenspinors can be derived directly by using the charge conjugation [3]. These are

$$V_{n,j,m_j} = (U_{n,j,m_j})^c \equiv C(\overline{U}_{n,j,m_j})^T \,, \quad C = i\gamma^2\gamma^0 \,. \tag{26}$$

Furthermore, we can verify that all these normalized energy eigenspinors have good orthogonality properties obeying

$$\begin{aligned} &\int_D d^3x\, \mu(\mathbf{x}) \overline{U}_{n,j,m_j}(\mathbf{x})\gamma^0 U_{n',j',m'_j}(\mathbf{x}) \\ &\qquad = \int_D d^3x\, \mu(\mathbf{x}) \overline{V}_{n,j,m_j}(\mathbf{x})\gamma^0 V_{n',j',m'_j}(\mathbf{x}) = \delta_{n,n'}\delta_{j,j'}\delta_{m_j,m'_j}\,, \end{aligned} \tag{27}$$

$$\begin{aligned} &\int_D d^3x\, \mu(\mathbf{x}) \overline{U}_{n,j,m_j}(\mathbf{x})\gamma^0 V_{n',j',m'_j}(\mathbf{x}) \\ &\qquad = \int_D d^3x\, \mu(\mathbf{x}) \overline{V}_{n,j,m_j}(\mathbf{x})\gamma^0 U_{n',j',m'_j}(\mathbf{x}) = 0\,, \end{aligned} \tag{28}$$

where

$$\mu(\mathbf{x}) = \frac{w(r)^3}{v(r)^2} = \frac{1}{\omega^2 r^2} \sin^2 \omega r \sec^3 \omega r \tag{29}$$

is the specific relativistic weight function [2]. Of course, the factors ω and $1/\omega^2$ can be removed simultaneously from (25) and respectively (29).

The final result is that for $M \ge \omega/2$, when only regular modes are allowed, the quantum Dirac field on CAdS reads

$$\psi(t,\mathbf{x}) = \sum_{n,j,m_j} \left[U_{n,j,m_j}(\mathbf{x}) e^{-iE_n t} a_{n,j,m_j} + V_{n,j,m_j}(\mathbf{x}) e^{iE_n t} b^{\dagger}_{n,j,m_j} \right]. \tag{30}$$

Our preliminary calculations indicate that the particle (a, $a^\dagger$) and antiparticle (b, $b^\dagger$) operators must satisfy usual anticommutation relations from which the non-vanishing ones are

$$\{a_{n,j,m_j}, a^\dagger_{n',j',m'_j}\} = \{b_{n,j,m_j}, b^\dagger_{n',j',m'_j}\} = \delta_{n,n'}\delta_{j,j'}\delta_{m_j,m'_j}. \tag{31}$$

The argument is that then the one-particle operators (Q, H, etc.) derived from the Noether theorem have similar structures and properties like those of the usual quantum field theory in flat spacetime. Hence we may have all the basic elements we need to construct the propagation theory of the regular modes of the Dirac field on CAdS, as the first step to a future theory of interacting quantum fields on this background.

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